Imaginary Numbers are Real!

Big Concepts:

- Number Systems
  How did number systems develop over the course of history? Why do complex/imaginary numbers exist?

- Complex Numbers
  Lesson ideas pulled from YouTube series on complex numbers: Imaginary Numbers are Real

Materials:

- Worksheet (attached)

- Classroom set of protractors (due to time constraints, it's better for each student to have their own rather than sharing.)

- Colored Pencils

Prep Before Lesson:

- Watch at least the first few segments of the 12-part series on Youtube (linked above).

- Make worksheet copies

Lesson Plan:

:00–:15 *Math Duel*

Divide students into 2 teams. Make an effort to try and make teams equal in terms of number of students from each grade level. Using the attached problems (or not), have students compete by solving for x before their opponent with no help (other than cheering) from their team. (The attached problems grow in difficulty from a 6th grade level to 9th grade level.) Get the kids excited. Once everyone has had a chance to go, congratulate the winners, and mention that although not so popular nowadays, Math Duels were a common occurrence in 16th century Italy. Bring in as much history as you want during this and the next bit.

:15–:30 *Number Systems*

Lead a discussion about the development of number systems. (Positive Naturals $\rightarrow$ Nonnegative Naturals $\rightarrow$ Integers $\rightarrow$ Rationals $\rightarrow$ Reals.) At each stage, ask what they think would logically come next and why.

:30–:45 *Development of the Complex Plane*

On the board, draw the real number line, plot the number 2, and ask, "where does i go?". Few may know, most may not. Don't let the few spoil it. Proceed through a dialogue similar to the following:
"What happens when we multiply by 1?"
"What happens when we multiply by 2?"
"What happens when we multiply by -1?" (draw a rotation of 180°)
"What if we multiply by -1 again?"
"So if we know that $i^2 = -1$, what happens when we multiply by i^2?
"Therefore, what do you guys think happens when we multiply by i? (They should be able to guess rotation by 90°)
"This is the complex plane!"

:45-1:30 *Self Discovery of Geometry from Complex Arithmetic*

Hand out the worksheet. Walk through the first adding problem on the board with them, including combining like terms and plotting the vectors in three different colors. Don't discuss observations yet. Let them do the other three in their groups. Then discuss the geometric effects of complex addition. Do the same for the multiplication section. Have them fill out the table on the last page. It is likely that once you finish discussion of this part, you will be out of time. If so, conclude. If not, proceed to writing in polar form section, which is straightforward.

These are sample problems for the math duels.

- $3x - 2 = -x - 6$
  $x = -1$

- $3(4x + 5) = 10x + 29$
  $x = 7$

- $4(3x - 2) + 3 = 31$
  $x = 3$

- $8x^3 + x^2 + 2 = 66 + x^2$
  $x = 2$

- $\frac{1}{3}(x - 2)^3 = 9$
  $x = 5$

- $(2x)^3 = -1$
  $x = -\frac{1}{2}$

- $\sqrt{x^2 + 300} = 20$
  $x = \pm 10$

- $x^3 + 8 = 0$
  $x = -2$

- $729 = x^3$
  $x = 9$

- $-5 = x^2 - 9x + 9$
  $x = 2, 7$

- $x^2 + 3x - 10 = 2x + 10$
  $x = 4, -5$

- $7x^2 + 9x - 3 = 2x^2 + 3x - 4$
  $x = -0.2, -1$ (quadratic formula)

- $(2x)^3 = 27$
  $x = \frac{9}{2}$

- $\frac{1}{2}(\sqrt{x} + 3) = \frac{11}{6}$
  $x = \frac{4}{9}$

- $x^3 - x^2 - 6x = 0$
  $x = -2, 0, 3$

- $x^2 + 2x + 4 = 2x + 3$
  $x = \pm i$

1 Do the Math: Complex Addition

Try adding up the following complex numbers. (Remember to only add like terms.) On the graphs below, plot each problem using three colors of your choosing.
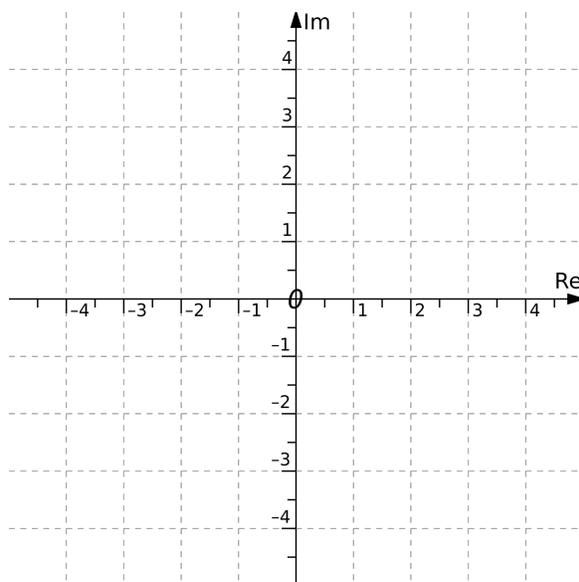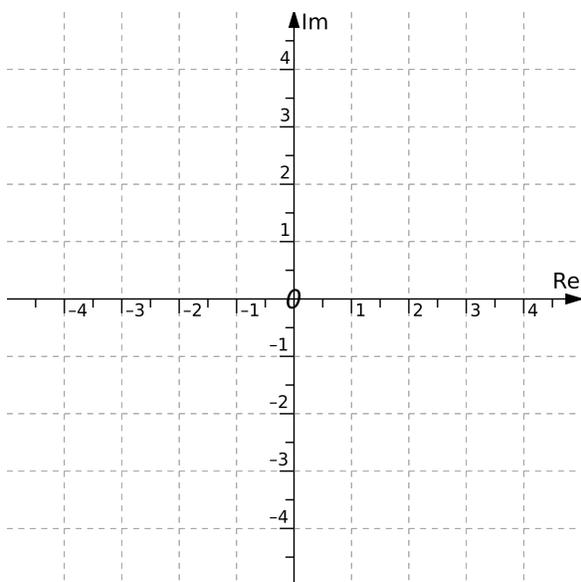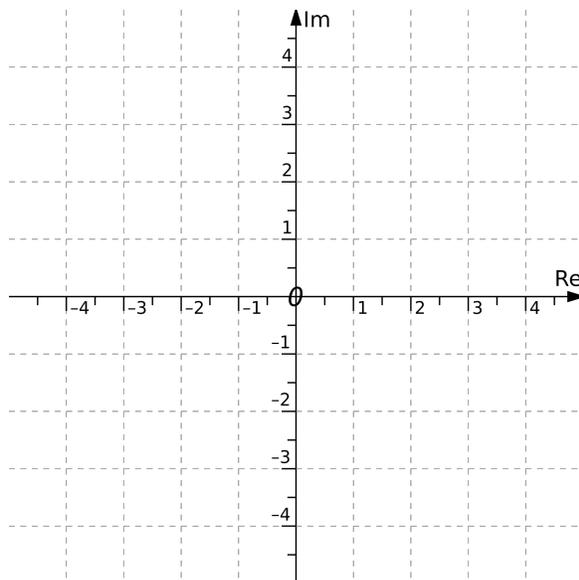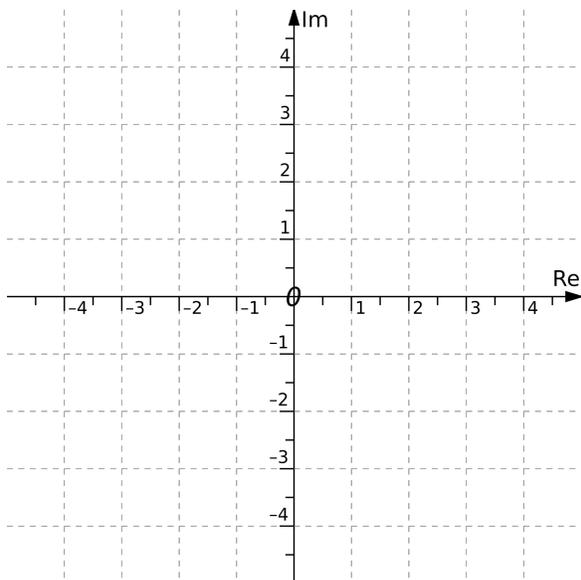
1. $(4 + i) + (-1 + 3i) =$

2. $(-2 + i) + (i) =$

3. $(2 - 2i) + (2 + 2i) =$

4. $(4 + 3i) - (1 + 2i) =$

2 Plot it!

3 Do the Math: Complex Multiplication

Just like we did with adding, try to multiply the following complex numbers, and plot them to see what happens. (Use your protractor to keep it neat and keep track of your angles.)
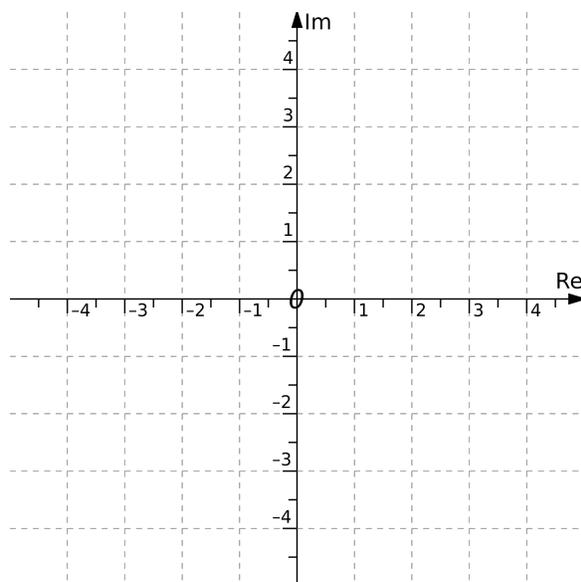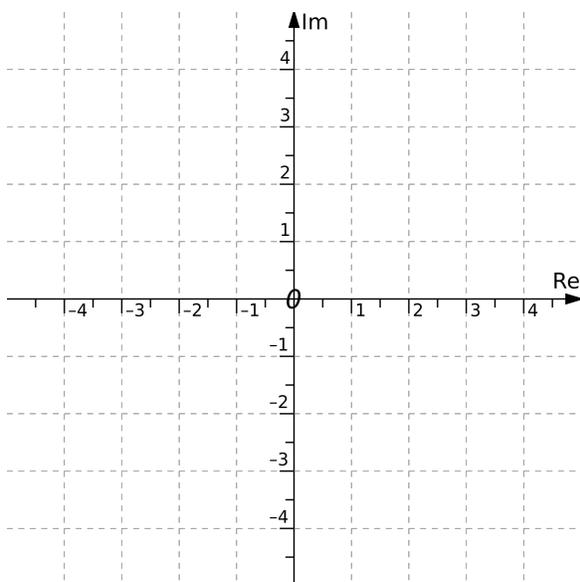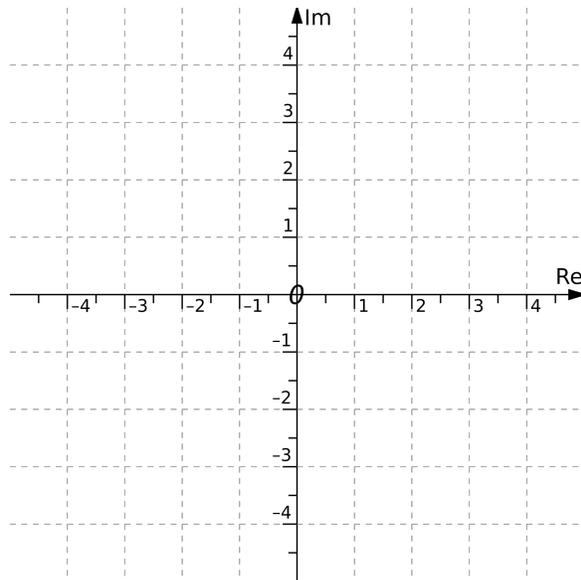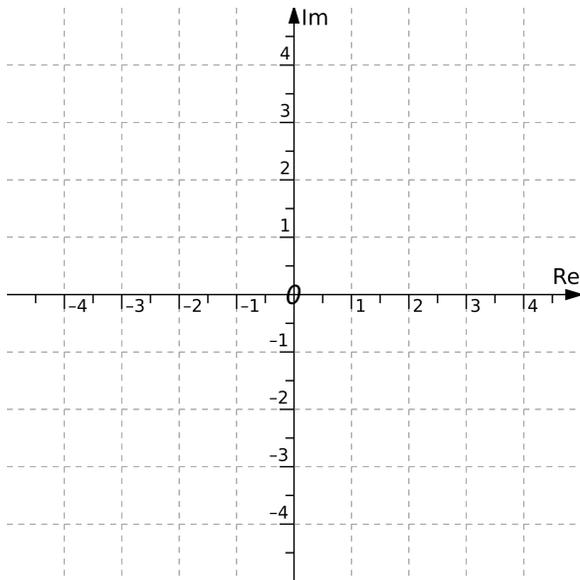
1. $(4 + 3i) \cdot i =$

2. $(2 + 2i) \cdot 2i =$

3. $(3 + i) \cdot (2 + i) =$

4. $(2 + i) \cdot (1 + 2i) =$

4 Plot it!

5   What Can We Discover?

Problem	Result	Angle 1	Angle 2	Result Angle	Dist 1	Dist 2	Result Distance
$(4 + 3i) \cdot i$							
$(2 + 2i) \cdot 2i$							
$(3 + i) \cdot (2 + i)$							
$(2 + i) \cdot (1 + 2i)$							

6   Alternative ways to write complex numbers to make it easier

As we all know, mathematicians are lazy. If something takes too long to write or solve, we are always looking for ways to make them easier. Because of this, we can write any complex number from before in what is known as **Polar Form**:

$$re^{i\theta}$$

where r represents the length (magnitude) of the line/vector and θ represents the angle (in radians) that the line makes with the positive real axis.

Try writing the following complex numbers in Polar Form:
(Hint: You should already have all the details you need for each of them)

1. $1 + i$

2. $2 + 2i$

3. $4 + 3i$

4. $-1 + 0i$